We'll start by discussing the formal error bound for Taylor polynomials. (I.e., how badly does a Taylor polynomial approximate a function?) Then, we'll see some standard examples. Finally, we'll see a powerful application of the error bound formula.

Lagrange Error Bound for $P_n(x)$

We know that the $n$th Taylor polynomial is $P_n(x)$, and we have spent a lot of time in this chapter calculating Taylor polynomials and Taylor Series. The question is, for a specific value of $x$, how badly does a Taylor polynomial represent its function? We define the error of the $n$th Taylor polynomial to be

$$E_n(x) = f(x) - P_n(x).$$

That is, error is the actual value minus the Taylor polynomial's value. Of course, this could be positive or negative. So, we force it to be positive by taking an absolute value.

$$|E_n(x)| = |f(x) - P_n(x)|.$$

The following theorem tells us how to bound this error. That is, it tells us how closely the Taylor polynomial approximates the function. Essentially, the difference between the Taylor polynomial and the original function is at most $|E_n(x)|$. At first, this formula may seem confusing. I'll give the formula, then explain it formally, then do some examples. You may want to simply skip to the examples.

**Theorem 10.1 Lagrange Error Bound** Let $f$ be a function such that it and all of its derivatives are continuous. If $P_n(x)$ is the $n$th Taylor polynomial for $f(x)$ centered at $x = a$, then the error is bounded by

$$|E_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$

where $M$ is some value satisfying $|f^{(n+1)}(x)| \leq M$ on the interval between $a$ and $x$.

**Explanation**

We derived this in class. The derivation is located in the textbook just prior to Theorem 10.1. The main idea is this: You did linear approximations in first semester calculus. What you did was you created a linear function (a line) approximating a function by taking two things into consideration: The value of the function at a point, and the value of the derivative at the same point. You built both of those values into the linear approximation. A Taylor polynomial takes more into consideration. It considers all the way up to the $n$th derivative. So, the first place where your original function and the Taylor polynomial differ is in the $(n+1)$st derivative. Really, all we're doing is using this fact in a very obscure way. We differentiated $n+1$ times, then figured out how much the function and Taylor polynomial differ, then integrated that difference all the way back $(n+1)$ times.
Basic Examples

1. Find the error bound for the 3rd Taylor polynomial of \( f(x) = \cos(x) \) centered at \( x = 0 \) on \([0, 2\pi]\).

**Solution:** This is really just asking “How badly does the 3rd Taylor polynomial to \( \cos(x) \) approximate \( \cos(x) \) on the interval \([0, 2\pi]\)?” Intuitively, we’d expect the Taylor polynomial to be a better approximation near where it is centered, i.e. near \( x = 0 \). We have

\[
|E_3(x)| \leq \frac{M}{4!} x^4
\]

where \( M \) bounds \( f^{(4)}(x) \) on the given interval \([0, 2\pi]\). But, we know that the 4th derivative of \( \cos(x) \) is \( \cos(x) \), and this has a maximum value of 1 on the interval \([0, 2\pi]\). So, we have \( M = 1 \). Thus, we have a bound

\[
|E_3(x)| \leq \frac{x^4}{4!}
\]

given as a function of \( x \). Now, if we’re looking for the worst possible value that this error can be on the given interval (this is usually what we’re interested in finding) then we find the maximum value that \( x^4 \) can take on the given interval. That maximum value is \((2\pi)^4 = 16\pi^4\). Hence, we know that the 3rd Taylor polynomial for \( \cos(x) \) is at least within

\[
\frac{16\pi^4}{4!} = \frac{2\pi^4}{3} \approx 64.94
\]

of the actual value of \( \cos(x) \) on the interval \([0, 2\pi]\).

2. What is the maximum possible error of the 100th Taylor polynomial of \( f(x) = e^x \) centered at \( x = 0 \) on the interval \([-10, 10]\)?

**Solution:** We have

\[
|E_{100}(x)| \leq \frac{M}{101!} x^{101}
\]

where \( M \) bounds \( f^{(101)}(x) = e^x \) on \([-10, 10]\). Since \( e^x \) takes its maximum value on \([-10, 10]\) at \( x = 10 \), we have \( e^{10} = M \). Thus, we have

\[
|E_{100}(x)| \leq \frac{e^{10}}{101!} x^{101}.
\]

What is the worst case scenario? When \( x^{101} \) is the largest is when \( x = 10 \). Thus, we have

\[
|E_{100}(x)| \leq \frac{e^{10} 10^{101}}{101!} \approx 2.3367 \times 10^{-55}.
\]

In other words, the 100th Taylor polynomial for \( e^x \) approximates \( e^x \) very well on the interval \([-10, 10]\).
A More Interesting Example

Problem: Show that the Taylor series for $\cos(x)$ is actually equal to $\cos(x)$ for all real numbers $x$.

Proof: The Taylor series is the “infinite degree” Taylor polynomial. So, we consider the limit of the error bounds for $P_n(x)$ as $n \to \infty$. That is, we’re looking at

$$\lim_{n \to \infty} |E_n(x)| \leq \lim_{n \to \infty} \frac{M}{(n+1)!} x^{n+1}.$$ 

Since all of the derivatives of $\cos(x)$ satisfy $-1 \leq \frac{d^n}{dx^n} \cos(x) \leq 1$, we know that $M = 1$. Thus, we have

$$\lim_{n \to \infty} |E_n(x)| \leq \lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{x^n}{n!}.$$ 

But, it’s an off-the-wall fact that

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0.$$ 

Thus, we have shown that

$$\lim_{n \to \infty} |E_n(x)| = 0$$

for all real numbers $x$. Thus, as $n \to \infty$, the Taylor polynomial approximations to $\cos(x)$ get better and better.