Permutation Group Algorithms, Part 2

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Two recent opening sentences for presentations on polynomial-time permutation group algorithms have each had five m's, one q, and one z, but this one is different in that last weeks didn't have thirteen a's, two b's, four c's, seven d's, forty-six e's, fifteen f's, five g's, thirteen h's, twenty-seven i's, one j, two k's, six l's, thirty n's, twenty o's, six p's, fifteen r's, forty s's, thirty-six t's, five u's, eleven v's, seven w's, five x's, and eight y's.

Resources:

- GAP code for Schreier-Sims functions (under this talk) at

  `http://math.jasonbhill.com/talks`

- Alexander Hulpke's "Notes on Computational Group Theory"
- Holt, et al's "Handbook of Computational Group Theory"
- Seress' "Permutation Group Algorithms"

1. Reminders

2. Stabilizer Chains and Strong Generating Sets

3. Backtrack

Reminders Of Where We Are

Last time, we reviewed the following as related to permutation groups:

- Group Actions
- Orbits and Point Stabilizers
- Various Definitions: Degree, Transitive, Primitive, Base

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- Orbit Algorithm with Transversal
- Orbit Algorithm with Schreier Vector
- Orbit Stabilizer Algorithm (as a consequence of Schreier's Theorem)
- Normal Closure (We didn't explain this entirely. In essence, we need some closure here. We'll assume for right now that this is possible in P.)

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- Derived and Lower Central Series
- Test if G is Solvable or Nilpotent

Reminders Of Where We Are

Goals For Today:

- Stabilizer Chains
- Schreier-Sims Algorithm
- Introduction to Backtrack

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Stabilizer Chain

A stabilizer chain for G with respect to B is the chain of subgroups defined by $G^{(0)} = G$, $G^{(1)} = G_{\beta_1} = \mathrm{Stab}_G(\beta_1)$, and

$$G^{(i)} = G_{\beta_i}^{G^{(i-1)}} = \mathrm{Stab}_{G^{(i-1)}}(\beta_i) = \mathrm{Stab}_G(\beta_1, \ldots, \beta_i)$$

where the last group may be viewed as a tuple (not set) stabilizer.

Stabilizer Chains

Example: Let $G = S_5$ act on $\Omega = [1 \dots 5]$ with $B = [1 \dots 4]$.

$$\begin{array}{ll}
 & G^{(0)} = S_5 \\
5 & \\
 & G^{(1)} = G_1^{(0)} = S_5|_{[2\dots5]} \simeq S_4 \text{ on } [2 \dots 5] \\
4 & \\
 & G^{(2)} = G_2^{(1)} = S_5|_{[3\dots5]} \simeq S_3 \text{ on } [3 \dots 5] \\
3 & \\
 & G^{(3)} = G_3^{(2)} = S_5|_{[4,5]} \simeq S_2 \text{ on } [4, 5] \simeq Z_2 \\
2 & \\
 & G^{(4)} = G_4^{(3)} = S_5|_{[5]} \simeq \langle\,\rangle
\end{array}$$

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- Schreier's Theorem yields (Schreier) generators for each $G^{(i)}$ ($i \geq 1$).

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- Inductively, each $g \in G$ has a (unique) decomposition

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- The transversal elements $\overline{b_i}$ (found via a Schreier vector) and corresponding orbit points, plus generators for the stabilizers at each level in the chain are given by the Orbit-Stabilizer Algorithm discussed last week.

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Schreier-Sims Algorithm

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Strong Generating System (SGS)

A generating set $\underline{g}$ is called a strong generating set for G relative to a base B, if $\langle \underline{g} \cap G^{(i)} \rangle = G^{(i)}$ for $0 \leq i \leq |B| - 1$. Hence, a stabilizer chain computed from an SGS will have transversal products as words in $\underline{g}$.

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Schreier-Sims Algorithm (Sims, 1970)

- **Input:** an arbitrary generating set $\underline{g}$, optional partial/full base.
- **Output:** BSGS

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Details: See the handout.

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- Construct a stabilizer chain. Instead of adding Schreier generators at each level, determine if those generators can be sifted with the existing chain's Schreier vectors. If not, add them to the SGS.

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- If the base is not long enough, the group will not factor entirely from the given chain and a new base element is needed.
- Whenever the calculated BSGS changes, initialize the stabilizer chain calculation again.

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This corresponds to the stabilizer chain:

Group	Generators	Orbit	Schreier Vector
$G^{(0)}$	$[(), (2,3), (1,2), (1,2,3)]$	$[1,2,3]$	$[(), 3, 2]$
$G^{(1)}$	$[(), (2,3)]$	$[2,3]$	$[(), 2]$
$G^{(2)}$	$[()]$	$[]$	$[()]$

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- We shall briefly discuss some improved versions of Schreier-Sims, and then consider consequences in the remainder of this talk.

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- $\mathrm{Alt}(\Omega)$ and $\mathrm{Sym}(\Omega)$ have known BSGS structures.
- If a giant group is found, then consider $\mathrm{sgn}(g)$ for $g \in \underline{g}$.

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- Picking enough random elements (≈ 10 with some uniform distribution) has a high probability of providing a SGS.

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- Given a permutation g we may test $g \in G$ by attempting to factor via coset representatives.
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- Obtain random elements with guaranteed equal distribution.

Example of Testing Group Membership

Group Membership in Dihedral Group of Degree 6

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[ (4,6,5), 3 ]  (Factorization Not Complete)
Fail
gap>
```

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- Recall that $B^g = [\beta_1^g, \ldots, \beta_k^g]$ uniquely determines each $g \in G$.

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- We will now organize such a search, known as **Backtrack**.

- Assume we have a BSGS $(B, \underline{g})$ for G.
- Recall that $B^g = [\beta_1^g, \ldots, \beta_k^g]$ uniquely determines each $g \in G$.
- First, we create an order relative to B^g.

Backtrack

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2. For $\alpha, \gamma \in \Omega \backslash B$, $\alpha \prec \gamma$ if $\alpha < \gamma$.
3. If $\beta \in B$ and $\alpha \in \Omega \backslash B$ then $\beta \prec \alpha$.

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- By definition of base, the identity element will always be least w.r.t $\prec$.

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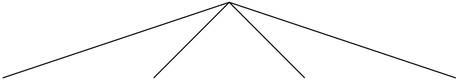
  - Base: `[ 1, 2, 5 ]`
- A few base images (by order of $\prec$):
  - `[ 1, 2, 5, 3, 4, 6 ]=()`
  - `[ 1, 2, 6, 3, 4, 5 ]=(5,6)`
  - `[ 1, 4, 5, 3, 2, 6 ]=(2,4)`
  - `[ 1, 4, 6, 3, 2, 5 ]=(2,4)(5,6)`

Searching Through Group Elements

Main Idea: Create a tree, where nodes represent transversal products and branches represent stabilizer cosets.



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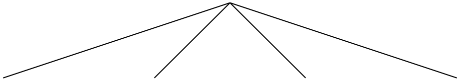

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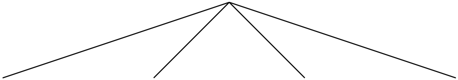

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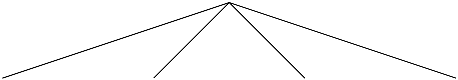

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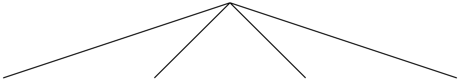

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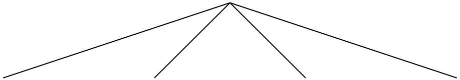

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- Order the other cosets from left to right based on the base image.
- The cosets of $G^{(2)}$ then give branches on the next level.

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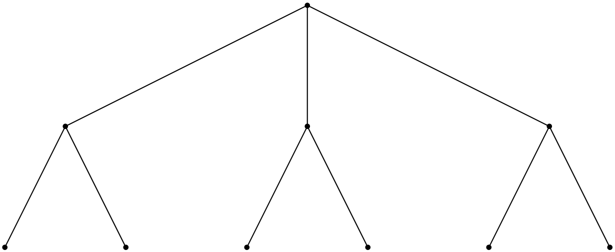

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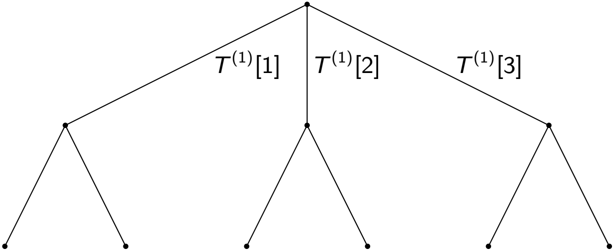

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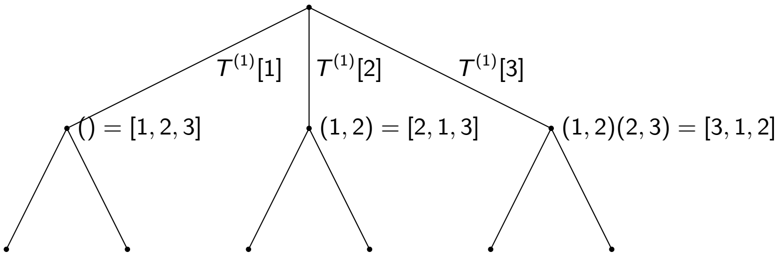

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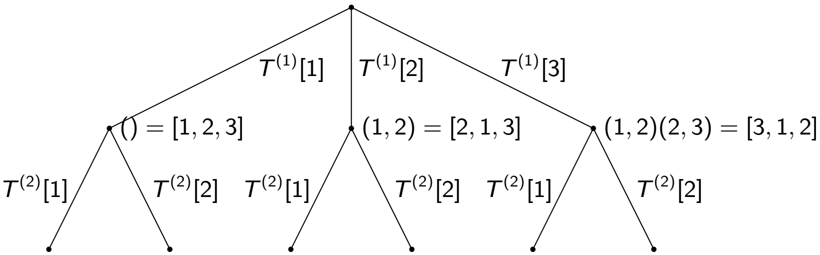

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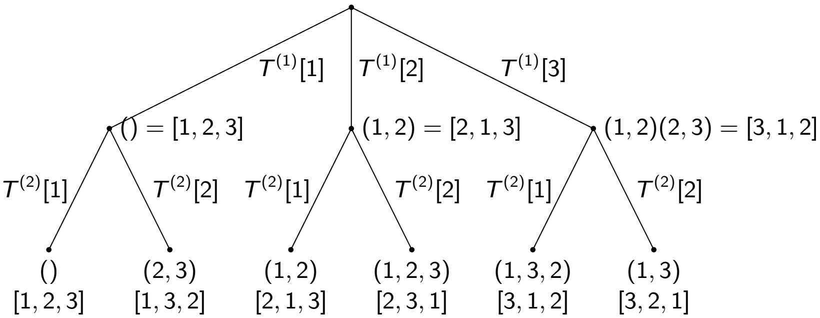

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Next week, we will consider this process in more depth and look at specific problems that reduce to this procedure. Also, we'll consider improvements using group structure.

Thank You