Permutation Group Algorithms, Part 1

Jason B. Hill

University of Colorado

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Slide one of this presentation by Jason B. Hill on polynomial time permutation group algorithms has a sentence containing ten a's, three b's, three c's, three d's, forty-one e's, nine f's, eight g's, fifteen h's, twenty-five i's, two j's, one k, eight l's, five m's, twenty-eight n's, twenty-one o's, five p's, one q, twelve r's, thirty-two s's, thirty-five t's, three u's, six v's, eight w's, two x's, nine y's, and one z.

Resources:

- GAP code for Schreier-Sims functions (under this talk) at

  `http://math.jasonbhill.com/talks`

- Alexander Hulpke's "Notes on Computational Group Theory"
- Holt, et al's "Handbook of Computational Group Theory"
- Seress' "Permutation Group Algorithms"

1. GAP and Sage

2. Background

3. Memory and Time

4. Permutation Group Algorithms in Polynomial Time

GAP and Sage

During this talk, I will reference two (free) software packages:

GAP – Groups, Algorithms and Programming

`http://www.gap-system.org/`

Sage (especially Sage Combinat)

`http://www.sagemath.org/`

`http://wiki.sagemath.org/combinat`

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Group([ (1,2,3), (1,2) ])
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Sage: (as a Python list of Python tuples)

```
sage:  G=PermutationGroup([(1,2,3),(1,2)])
sage:  G.order()
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GAP –vs– Sage

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- Notice that this makes a notion of primitivity in Sage, for instance, not precise with existing literature. (What is the size of a block?)
- Recent patches to Sage Combinat by Mike Hansen and myself have made Sage more consistent with GAP, but GAP is still better suited to serious work.

Background

Group Actions

Group Action

A group G acts on a set (domain) Ω if

- $\omega^1 = \omega$ for all $\omega \in \Omega$.
- $(\omega^g)^h = \omega^{gh}$ for all $\omega \in \Omega$ and all $g, h \in G$.

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- The group operation is composition of permutations.
- The same group may be used in vastly different actions. Upon fixing an action, we will refer to both the group and the action as G. (We will see an example shortly of why such things matter.)

Orbits and Point Stabilizers

Orbits and Point Stabilizers

For $\omega \in \Omega$ define the orbit

$$\omega^G = \{\omega^g \mid g \in G\} \subset \Omega$$

and the point stabilizer

$$G_\omega = \mathrm{Stab}_G(\omega) = \{g \in G \mid \omega^g = \omega\}.$$

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Orbit-Stabilizer Theorem: For $\omega \in \Omega$ there is a bijection between ω^G and the set $G_\omega{}^{\backslash G}$ (right cosets of G_ω in G) given by $\omega^g \leftrightarrow G_\omega \cdot g$. In particular (for finite groups anyway) $|\omega^G| = [G : G_\omega]$.

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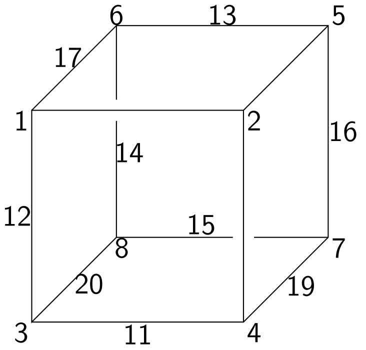
Base

A base for G is a subset $B \subset \Omega$ such that the only element of G stabilizing all points of B is the identity.

Cube Rotations

Example: Consider the rotational group of the cube:

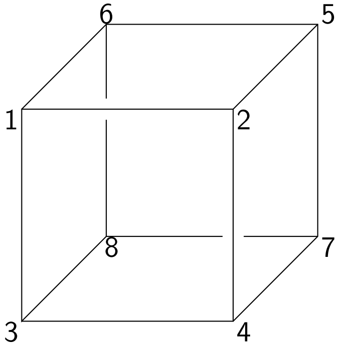
```
gap> x:=(1,2,4,3)(5,7,8,6)(9,10,11,12)(13,16,15,14)(17,18,19,20);;
gap> y:=(1,6,8,3)(2,5,7,4)(9,13,15,11)(10,18,16,19)(12,17,14,20);;
gap> z:=(1,2,5,6)(3,4,7,8)(9,18,13,17)(10,16,14,12)(11,19,15,20);;
gap> G:=Group(x,y,z);;
```



Cube Rotations

We could consider acting only on the vertices:

```
gap> r:=(1,2,4,3)(5,7,8,6);;
gap> s:=(1,6,8,3)(2,5,7,4);;
gap> t:=(1,2,5,6)(3,4,7,8);;
gap> H:=Group(r,s,t);;
```



Cube Rotations

```
gap> IsomorphismGroups(G,SymmetricGroup(4));
gap> IsomorphismGroups(H,SymmetricGroup(4));
```

show us that

$$\varphi_1 : G \to S_4 : \begin{cases} x \mapsto (1,2,3,4) \\ z \mapsto (1,4,2,3) \end{cases} \quad \text{and} \quad \varphi_2 : H \to S_4 : \begin{cases} r \mapsto (1,2,3,4) \\ t \mapsto (1,4,2,3) \end{cases}$$

are isomorphisms. Hence, $G \simeq H \simeq S_4$ while the corresponding actions are clearly different.

Cube Rotations Summary

	$G \leq S_{20}$	$H \leq S_8$	S_4
domain	$[1 \ldots 20]$	$[1 \ldots 8]$	$[1 \ldots 4]$
transitive	no	yes	yes
orbits	$[1 \ldots 8]$ and $[9 \ldots 20]$	$[1 \ldots 8]$	$[1 \ldots 4]$
primitive	no	no	yes
block system	edges and vertices	long diagonals	trivial
base	$[1, 2]$	$[1, 2]$	$[1, 2, 3]$

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Moral: Intuitive understandings of S_4 can differ drastically from its real-world implementations. Such situations need to be taken into account when designing data structures or algorithms that handle these groups. If we want to work with groups having degree $> 10^6$, for instance, then intuition probably won't help guide us much. But, if we work with a well-known group, the output should make sense.

Memory and Time

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While our intuitive understanding of groups/actions may place extraneous requirements on a data structure or algorithm, there are two restrictions that mechanically limit even a perfect data structure or algorithm design.

1. Memory
2. Computation Time

Memory

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- Thus, storing base images may be a reasonable approach.
- For example, in S_5 with base $[1, 2, 3, 4]$, the permutation $(1, 2)$ would be recorded as $[2, 1, 3, 4]$ and $(1, 5)(2, 3, 4)$ as $[5, 3, 4, 2]$.

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- For example, in S_5 with base $[1, 2, 3, 4]$, the permutation $(1, 2)$ would be recorded as $[2, 1, 3, 4]$ and $(1, 5)(2, 3, 4)$ as $[5, 3, 4, 2]$.
- Limit parentheses, extra characters, etc.

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- A base for S_n is $[1, \ldots, n-1]$, so each base image is expressed in

$$\sum_{j=1}^{n-1} \lceil \log_2(j) \rceil$$

  bits. (This is of course a conservative estimate.) Multiplying by $n!$ and dividing by 8×10^{12} we find that all elements of S_n are recordable in no less than

$$\frac{\log_2\left((n-1)!^{n!}\right)}{8 \times 10^{12}} = \frac{\log_2\left(\Gamma(n)^{\Gamma(n+1)}\right)}{8 \times 10^{12}} \text{ terabytes.}$$

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- In this manner, turning every atom in the observable universe into a 1TB hard disk allows one to only consider up to S_{64}.
- Clearly we need a different approach.

Polynomial Time

$$L \subseteq AL = P \subseteq NP \subseteq PSPACE \subseteq EXPTIME$$

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Examples:

- The "quicksort" sorting algorithm on n integers performs at most An^2 operations for some constant A. Thus it runs in time $O(n^2)$.
- All basic arithmetic operations on a computer can be done in polynomial time.

Permutation Group Algorithms in Polynomial Time

"Plain Vanilla" Orbit Algorithm

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“Plain Vanilla” Orbit Algorithm

Input: $\underline{g} = \{g_1, \ldots, g_m\}$, $\omega \in \Omega$. **Output:** ω^G.

```
$\Delta := [\omega]$;
for $\delta \in \Delta$ do
    for $i \in \{1, \ldots, m\}$ do
        $\gamma := \delta^{g_i}$;
        if $\gamma \notin \Delta$ then
            Append $\gamma$ to $\Delta$;
        fi;
    od;
od;
return $\Delta$;
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“Plain Vanilla” Orbit Algorithm

Example: Using the group $G \leq S_{20}$ (cube rotations) from earlier:

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gap> jbhOrbitAlgVanilla([x,y,z],2);
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- Thus, naively, the algorithm has a linear runtime.
- However, note that step 2 is non-primitive recursive.
- Also, step 5 is a search problem.
- Once the orbit is created, information about specifically how it was created is lost. This would be nice information to store and introduces little complexity.

Modified Orbit Algorithm

We will change steps 1 and 6 of the algorithm:

Orbit Algorithm

Input: $\underline{g} = \{g_1, \ldots, g_m\}$, $\omega \in \Omega$. **Output:** ω^G, transversal T.

```
1: $\Delta := [\omega]$, $T = [()]$;
2: for $\delta \in \Delta$ do
3:     for $i \in \{1, \ldots, m\}$ do
4:         $\gamma := \delta^{g_i}$;
5:         if $\gamma \notin \Delta$ then
6:             Append $\gamma$ to $\Delta$, Append $T[\delta] \cdot g_i$ to $T$;
7:         fi;
8:     od;
9: od;
10: return $\Delta$;
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Modified Orbit Algorithm

Example:

```
gap> jbhOrbitAlgTrans([(1,2,3),(4,5,6)],2);
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- By the Orbit-Stabilizer Theorem, T is simultaneously a set of representatives for the cosets of $G_\omega = \mathrm{Stab}_G(\omega) \leq G$.
- Almost for free, we get limited information about stabilizer subgroups. We should exploit this idea more.

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Schreier Vector for $\Delta = \omega^G$

A Schreier vector is a list $S = [S[\Delta[1]], S[\Delta[2]], \ldots, S[\Delta[|\Delta|]]]$ satisfying

1. $S[\Delta[i]] \in \underline{g}$ for $1 \leq i \leq |\Delta|$.
2. $S[\omega] = ()$
3. $S[\delta] = g$ ad $\delta^{g^{-1}} = \gamma$, then γ precedes δ in Δ.

Sometimes a Schreier vector is called a “factored transversal.”

Orbit Algorithm with Schreier Vector

Orbit Algorithm with Schreier Vector

Input: $\underline{g} = \{g_1, \dots, g_m\}$, $\omega \in \Omega$. **Output:** ω^G, Schreier vector S.

```
$\Delta := [\omega]$, $S = [1]$;
for $\delta \in \Delta$ do
    for $i \in \{1, \dots, m\}$ do
        $\gamma := \delta^{g_i}$;
        if $\gamma \notin \Delta$ then
            Append $\gamma$ to $\Delta$, Append $i$ to $S$;
        fi;
    od;
od;
return $\Delta$;
```

Schreier Vectors

Example:

```
gap> jbhOrbitAlgTrans([(1,2,3),(4,5,6)],2);
[ [ 2, 3, 1 ], [ (), (1,2,3), (1,2,3)(1,2,3) ] ]
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Question: We now have a way to find coset representatives of point stabilizers in linear time. Can this be translated in some way to yield generators for the point stabilizers? (Yes.)

Schreier's Theorem

Schreier's Theorem

Let $G = \langle \underline{g} \rangle$ be a finitely generated group and $H \leq G$ with $[G : H] < \infty$. Suppose $\underline{r} = \{r_1 = 1, r_2, \ldots, r_m\}$ is a set of (right) coset representatives for H in G. For $k \in G$ write $\overline{k}$ to denote the representative $\overline{k} := r_i$ with $Hr_i = Hk$. Let

$$U := \{r_i g_j \overline{(r_i g_j)}^{-1} \mid r_i \in \underline{r},\ g_j \in \underline{g}\}.$$

Then $H = \langle U \rangle$. U is called a set of Schreier generators for H.

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Question: Can we use this process to decompose an entire group, recursively by one point-stabilizer at a time? (Yes.)

A Slight Detour: Normal Closure

Normal Closure

Let $U \leq G$. The normal closure of U in G is

$$\langle U \rangle_G = \bigcap \{ N \mid U \leq N \lhd G \}$$

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Without too much work, the orbit algorithm can be modified to calculate normal closure. (Instead of acting on domain points by permutations, we act on group elements by conjugation.)

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$$G' = \left\langle a^{-1} b^{-1} ab \mid a, b \in \underline{g} \right\rangle_G$$

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- Computation of the derived series and lower central series.
- Is G solvable? Is G nilpotent?

Thank You